STUDY OF THE PERMANENT CONJECTURE AND SOME OF ITS GENERALIZATIONS †

BY

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ABSTRACT

In this paper, for some of the results which were announced in [3], we define, for every convex polytope, a function generalizing the permanent; we study the growth of the function, the behavior of its minimum and determine a lower bound for the minimum. Some of the results are new even for the permanent. A related function whose properties are strongly linked to the permanent is also studied, and will be described in greater detail in a sequel to this paper.

For any set T in a real affine space A, we denote by $\bar{T}$ the least affine subspace containing T. Let K be a non-empty convex polytope in A. By $\underline{K}$ we denote the interior of K with respect to $\bar{K}$. We shall also speak of the relative boundary of K, which is defined with respect to $\bar{K}$. The dimension of K shall be the dimension of $\bar{K}$.

Let $S = \{A_1, A_2, \cdots, A_n\}$ be a collection of (not necessarily distinct) affine functions on A. We say that S is a determining set for K if $K = \{x \in \bar{K} \mid A_i(x) \geqq 0 \, \forall i\}$. If S is a determining set for K, and we have for two points x and y in $\bar{K}$ that $A_i(x) = A_i(y)$ for all i, then necessarily $x = y$. For let $u \in K$, then for any scalar λ,

$$A_i(u + \lambda(y - x)) = A_i(u) + \lambda(A_i(y) - A_i(x)) = A_i(u) \geqq 0.$$

Since K is compact, we must have $y = x$.

Now let $\mathscr{E}$ be a finite set of points in A, let K be the convex hull of $\mathscr{E}$, and let S be a determining set for K. $\mathscr{E}$ contains, then, the set $\mathscr{E}'$ of extreme points of K. For the sake of convenience, we are going to assume throughout this paper that no A_i vanishes identically on K. For any subset $\{i_1, i_2, \cdots, i_v\}$ of $\{1, 2, \cdots, n\}$, we define

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$$F(i_1, i_2, \cdots, i_v) = \{x \in K \mid A_i(x) = 0, \qquad i = i_1, i_2, \cdots, i_v\}$$

and call it the flat determined by $\{i_1, i_2, \cdots, i_v\}$. Every flat F is a convex polyhedron whose set of extreme points is precisely $F \cap \mathscr{E}'$, and S is also a determining set for F. The flat $F = F(i_1, i_2, \cdots, i_v)$ is said to be regularly defined if there exists $x \in F$ such that $A_i(x) > 0$ for $i \neq i_1, i_2, \cdots, i_v$. Clearly every non-empty flat may be regularly defined. As is usual, a flat of codimension one in K will be called a face of K.

Let R^n_0 be the closed first 2^n-gant in R^n, R^n_+ its interior. R^n_0 is a multiplicative semigroup under multiplication per coordinate, and the product of u and v will simply be denoted uv. We define additionally $u^v = u_1^{v_1} u_2^{v_2} u_n^{v_n}$, $0^0 = 1$. If α is a scalar greater than or equal to zero, we define $u^\alpha = (u_1^\alpha, u_2^\alpha, \cdots, u_n^\alpha)$. If $u \in R^n_+$, we define u^α as above for any scalar α. Also for $\alpha \geqq 0$ and $u \in R^n_0$, we define $\alpha^u = (\alpha^{u_1}, \alpha^{u_2}, \cdots, \alpha^{u_n})$.

Returning now to the convex polytope K, we construct an affine map $A: K \to R^n_0$ by setting $A(x) = (A_1(x), A_2(x), \cdots, A_n(x))$. Let c be a strictly positive function on $\mathscr{E}$. For $y \in R^n_0$, define

$$Q(y) = \sum_{e \in \mathscr{E}} c(e) y^{A(e)}$$

and for $x \in K$, define

$$P(x) = Q(A(x)).$$

$P(x)$ is strictly positive, and we will study its minimum value, which may be related in fact to the permanent conjecture [5]. For if we let K be the set D_k of $k \times k$ doubly stochastic matrices sitting in the affine space R^{k^2}, take for S the set of coordinate functions, for $\mathscr{E}$ the permutation matrices, and $c \equiv 1$, then $P(x)$ is the permanent of x, $\operatorname{perm}(x) = \sum x^\pi$, where the summation is over the $k \times k$ permutation matrices.

Returning again to the general case, we define a map $q: R^n_0 \to K$, by setting

$$q(y) = \frac{1}{Q(y)} \sum_{e \in \mathscr{E}} c(e) y^{A(e)} e,$$

well defined for y such that $Q(y) \neq 0$, and a map $h: K \to K$, by setting

$$h(x) = q(A(x)).$$

A few remarks about the map h are in order. Note that if $A_i(x) = 0$, then trivially $A_i(h(x)) = 0$. But the converse is also true. For suppose $A_i(h(x)) = 0$. Then if for $e \in \mathscr{E}$, we have $A_i(e) \neq 0$, we must have $A(x)^{A(e)} = 0$. Now write $x = \sum_{\nu=1}^{k} p_\nu e_\nu$ as a proper ($p_\nu > 0\ \forall\nu$) convex combination of extreme points. Since $A(x) = \sum p_\nu A(e_\nu)$, we have $A(x)^{A(e_\nu)} \neq 0$ for $\nu = 1, 2, \cdots, k$, and so we must have $A_i(e_\nu) = 0$, implying $A_i(x) = 0$. All this means, of course, that h maps the relative interior of any flat into itself. From this it follows that h fixes the extreme points of K, which is immediate from inspection also.

Our first main result is Theorem 1.

THEOREM 1. *h is a bijection.*

The proof will be composed of a series of lemmas.

LEMMA 2. *h is onto.*

Let $F = F(r+1, r+2, \cdots, n)$ be a regularly defined flat, and let v and x belong to the relative interior of F. Consider the function of x given by

$$G(x) = \frac{P(x)}{A(x)^{A(v)}}.$$

As x approaches the relative boundary of F, $G(x)$ goes to plus infinity. Hence $G(x)$ has a minimum at a point $x_0 \in \underline{F}$. Put $w = h(x_0)$, $x = x_0 + t(w - v)$, and consider the function

$$g(t) = G(x) = G(x_0 + t(w - v))$$

defined for sufficiently small t and having a minimum at $t = 0$. A simple calculation shows that $(d/dt)A_i(x) = A_i(w) - A_i(v)$. We may now write $\log g(t) = \log P(x) - \sum_{i=1}^{r} A_i(v) \log A_i(x)$. Hence:

$$\begin{aligned} 0 = \frac{g'}{g}(0) &= \frac{1}{P(x_0)} \sum_{e \in \mathscr{E}} c(e) A(x_0)^{A(e)} \sum_{i=1}^{r} \frac{A_i(e)}{A_i(x_0)}(A_i(w) - A(v)) \\ &\qquad - \sum_{i=1}^{r} \frac{A_i(v)}{A_i(x_0)}(A_i(w)) - A_i(v)) \\ &= \sum_{i=1}^{r} \frac{A_i(w)}{A_i(x_0)}(A_i(w) - A_i(v)) - \sum_{i=1}^{r} \frac{A_i(v)}{A_i(x_0)}(A_i(w) - A_i(v)) \\ &= \sum_{i=1}^{r} \frac{(A_i(w) - A_i(v))^2}{A_i(x_0)} \end{aligned}$$

from which it follows that $A_i(w) = A_i(v)$ for all i, which implies that $h(x_0) = v$. Since any $v \in K$ belongs to the relative interior of some flat, the proof is completed.

Let μ be a not identically zero Borel measure of compact support in R^m equipped with the usual scalar product $\langle \cdot, \cdot \rangle$, and put

$$L(x) = \int \exp\langle x,s\rangle \mu(ds),$$

defined and differentiable for all $x \in R^m$ [4].

LEMMA 3. *$\partial^2 \log L(x)/\partial x_i \partial x_j$ is positive semi-definite and $\log L(x)$ is a convex function.*

As a straightforward computation shows:

$$L^2(x) \sum_{i,j} \frac{\partial^2 \log L(x)}{\partial x_i \partial x_j} r_i r_j = L^2(x) \frac{\partial^2}{\partial \lambda^2} \log L(x+\lambda r) \ (\lambda = 0)$$

$$= 1/2 \int \langle r, t-s\rangle^2 \exp\langle x, s+t\rangle \mu(ds)\mu(dt)$$

for any $r \in R^m$, which proves the lemma.

LEMMA 4. *Suppose* grad log $L(x)$ = grad log $L(y)$. *Then* $\langle y-x, s\rangle = \langle y-x, t\rangle$ *for any pair of points t and s in the support of μ.*

Let $r \in R^m$, $\lambda \in R$. As the last lemma shows,

$$\langle \mathrm{grad} \log L(x+\lambda r), r\rangle = \frac{d}{d\lambda} \log L(x+\lambda r)$$

is a non-decreasing function of λ. Put $r = y - x$, and compare at $\lambda = 0$ and $\lambda = 1$. Thus $\langle \mathrm{grad}\log L(y),\ y-x\rangle \geqq \langle \mathrm{grad}\log L(x),\ y-x\rangle$. Since we have equality in the last inequality, we must have that $\langle \mathrm{grad}\log L(x+\lambda r),\ r\rangle$ is constant for $0 \leqq \lambda \leqq 1$. Hence

$$0 = \frac{d^2}{d\lambda^2} \log L(x+\lambda r) \ (\lambda = 0)$$

$$= \frac{1}{2} \int \langle y-x,\ t-s\rangle^2 \exp\langle x,\ s+t\rangle \mu(ds)\mu(dt).$$

Since the integrand is non-negative, the conclusion of the lemma follows.

Now let $y \in R_0^n$. The support of y is defined to be the set of $1 \leqq i \leqq n$ such that $y_i \neq 0$. Let $F = F(r+1, r+2, \cdots, n)$ be a regularly defined flat. y is said to have F-like support if $y_1, y_2, \cdots, y_r \neq 0$, and $y_{r+1}, y_{r+2}, \cdots, y_n = 0$. This is the same as saying that y has the same support as $A(x)$ for some $x \in \underline{F}$. Clearly, if the support of y contains the support of $A(x)$ for any $x \in K$, then $Q(y) \neq 0$.

Let y have F-like support and put $u = (\log y_1, \log y_2, \cdots, \log y_r)$, $u \in R^r$. Define $B(e) = (A_1(e), A_2(e), \cdots, A_r(e))$ and put

$$L(u) = \sum_{e \in \mathscr{E} \cap F} c(e) \exp\langle u, B(e)\rangle.$$

Notice that $L(u)$ is the Laplace transform of a finitely supported measure, namel one having mass $c(e)$ at the point $B(e)$ for all $e \in \mathscr{E} \cap F$. Observe as well that $L(u) = Q(y)$, and that $(\operatorname{grad}\log L(u))_j = A_j(q(y))$, for $j = 1, 2, \cdots, r$.

LEMMA 5. *Let y and y' have F-like support, and suppose $q(y) = q(y')$. Then there exists $\alpha \in R^r$ such that $\sum_{i=1}^r \alpha_i A_i(x) = d$(constant) for all $x \in F$, and such that $y_i = e^{\alpha_i} y_i'$, $i = 1, 2, \cdots, r$.*

Construct u and u' related to y and y' as above. Then we have $\operatorname{grad}\log L(u) = \operatorname{grad}\log L(u')$. By Lemma 4, it follows that $\langle u - u', B(e)\rangle = \langle u - u', B(f)\rangle$ for any e and $f \in \mathscr{E} \cap F$; or $\sum_{i=1}^r \log(y_i/y_i')A_i(e) = d$. Put $\alpha_i = \log(y_i/y_i')$. Then $\sum_{i=1}^r \alpha_i A_i(e) = d$ for all $e \in \mathscr{E} \cap F$, and since every $x \in F$ may be written as a convex combination of such e, the lemma follows.

Notice that if y and y' are related as concluded in the lemma, then necessarily $q(y) = q(y')$.

LEMMA 6. *h is injective.*

For let x and x' be points in $\underline{F}$ such that $h(x) = h(x')$, that is, $q(A(x)) = q(A(x'))$. Then there exists $\alpha \in R^r$ satisfying the conclusions of Lemma 5, that is,

$$A_i(x) = e^{\alpha_i} A_i(x'), \quad i = 1, 2, \cdots, r, \quad \text{and} \quad \sum_{i=1}^{r} \alpha_i A_i(y) = d$$

for all $y \in F$. Put $a = \sum_{i=1}^r A_i(x)$ and $a' = \sum_{i=1}^r A_i(x')$. Then

$$\frac{a}{a'} = \sum_{i=1}^{r} \frac{A_i(x)}{a'} = \sum_{i=1}^{r} \frac{A_i(x')}{a'} e^{\alpha_i} \geqq \exp \sum_{i=1}^{r} \frac{A_i(x')}{a'} \alpha_i = e^{d/a'},$$

using convexity of the exponential function. Next, reverse the roles of x and x', replacing α and d by their negatives, to obtain $a'/a \geqq e^{-d/a}$. In the case $d = 0$, compare the two inequalities to obtain $a = a'$. But then the inequality $a/a' \geqq e^{d/a'}$ is an equality, and since the exponential function is strictly convex, we must have that α_i is a constant independent of i, and since $d = 0 = \sum_i \alpha_i A_i(y)$, α_i is in fact always zero.

In case $d > 0$, multiply the two inequalities to obtain $1 \geqq \exp(d(1/a' - 1/a))$, which implies $a' \geqq a$. But then $1 \geqq a/a' \geqq e^{d/a'}$, which is a contradiction. The case $d < 0$ is settled symmetrically.

So we must have $d = 0$, $\alpha_i = 0$, and then $A_i(x) = A_i(x')$ for $i = 1, 2, \cdots, r$, but trivially for $i = r+1, r+2, \cdots, n$, implying $x = x'$.

Since $h: K \to K$ is a bijection, it has an inverse which we denote by l. l is continuous by a well-known theorem on invertible maps of compact Hausdorff spaces. Actually much more is true.

THEOREM 7. *Let F be a flat (possibly K). Then l is a real analytic map of $\underline{F}$ to itself.*

Since h is a real analytic map of $\underline{F}$ to itself, l will be real analytic at points where the differential of h is non-singular. So let $F = F(r+1, r+2, \cdots, n)$ be regularly defined, $x \in \underline{F}$ and $v \in F$. Then the differential of h at the point x in the direction $v - x$ of the tangent space to F at x is given by $(d/dt)h(x + t(v - x))$ $(t = 0)$, and we must show this is zero only for $v = x$. A routine computation shows that the above is equal to

$$\frac{1}{P(x)} \sum_{e \in \mathscr{E}} c(e)A(x)^{A(e)} \left\{ \sum_{i=1}^{r} \frac{A_i(e)}{A_i(x)} (A_i(v) - A_i(x)) \right\} e - h(x) \sum_{i=1}^{r} \frac{A_i(h(x))}{A_i(x)} (A_i(v) - A_i(x)).$$

Suppose the above expression is zero. Noting that $(1/P(x)) \sum_{e \in \mathscr{E}} c(e)A(x)^{A(e)}A(e) = A(h(x))$, we obtain that

$$\frac{1}{P(x)} \sum_{e \in \mathscr{E}} c(e)A(x)^{A(e)} \left\{ \sum_{i=1}^{r} \frac{A_i(e)}{A_i(x)} (A_i(v) - A_i(x)) \right\} A_j(e)$$
$$- A_j(h(x)) \sum_{i=1}^{r} \frac{A_i(h(x))}{A_i(x)} (A_i(v) - A_i(x)) = 0 \text{ for any } j.$$

Let $s \in R^r$ be point whose ith coordinate is $(A_i(v) - A_i(x))/A_i(x)$. Then from the above we obtain further

$$\frac{1}{P(x)} \sum_{e \in \mathscr{E}} c(e)A(x)^{A(e)} \sum_{1 \leqq i,j \leqq r} A_i(e) A_j(e) s_i s_j$$
$$- \sum_{1 \leqq i,j \leqq r} A_i(h(x)) A_j(h(x)) s_i s_j = 0.$$

Now put $u = (\log A_1(x), \log A_2(x), \cdots, \log A_r(x))$, and form the function $L(u)$ as in Lemma 4. Another routine computation shows that the expression on the left immediately above is precisely $(d^2/d\lambda^2)\log L(u + \lambda s)$ $(\lambda = 0)$. But the vanishing of the last expression implies, referring to Lemma 3, that $\langle s, B(e) \rangle = d$ (constant) for all $e \in \mathscr{E} \cap F$, or $\sum_{i=1}^{r} [(A_i(v) - A_i(x))/A_i(x)] A_i(e) = d$, from which we obtain $\sum_{i=1}^{r} [(A_i(v) - A_i(x))/A_i(x)] A_i(w) = d$ for any $w \in F$. On comparing for $w = v$ and for $w = x$, we see that $\sum_{i=1}^{r} [(A_i(v) - A_i(x))^2/A_i(x)] = 0$, which implies that $v = x$.

We can also obtain a simple inequality for $P(x)$.

LEMMA 8. *For all x and $v \in K$, $P(x)/A(x)^{A(v)} \geqq P(l(v))/A(l(v))^{A(v)}$.*

First, let x and v belong to $\underline{K}$. Referring to Lemma 2, we have that the minimum (as a function of x) of $P(x)/A(x)^{A(v)}$ is achieved uniquely at point $x = l(v)$. Hence in this case $A(x)^{A(v)}/P(x) \leqq A(l(v))^{A(v)}/P(l(v))$. Since $x \in \underline{K}$, the left-hand side of the above is continuous in v. It is easy to see that $A(v)^{A(h(v))}/P(v)$ is continuous in v, so the right-hand side is likewise continuous. Thus the inequality holds for any $v \in K$ provided $x \in \underline{K}$. But now the left-hand side is a continuous function of $x \in K$, so the inequality holds generally.

Under certain further assumptions, the map h has an additional remarkable property. The function $Q(y)$ is homogeneous of degree d if $\sum_{i=1}^n A_i(e) = d$ for all $e \in \mathscr{E}$, which is same as saying $\sum_{i=1}^n A_i(x) = d$ for all $x \in \mathring{K}$. If, for example, K has a non-empty interior with respect to A, and A is equipped with a Euclidean metric, then since the sum of the inner normals to the faces of K with lengths equal to the area of the faces is zero, homogeneity can be achieved by appropriate choice of S. For the remainder of this paper, we will assume Q is homogeneous of degree d. In this case we have Theorem 9.

THEOREM 9. *$P(h(x)) \geqq P(x)$, with equality only for $h(x) = x$.*

The proof will follow from Lemma 10.

LEMMA 10. *Let $F(x)$ be a positive, twice continuously differentiable function on R^n_+ such that $\log F(e^x)$ is a convex function of x. Then for u and $v \in R^n_+$, $F(uv)/F(u) \geqq v^{u\, grad\, F(u)/F(u)}$.*

Put $G(x) = \log F(e^x)$. Then by Taylor's theorem

$$G(x+y) = G(x) + \text{grad } G(x), y\rangle + \frac{1}{2}\sum_{i,j} \frac{\partial^2 G}{\partial x\, \partial x_j}(x+\theta y) y_i y_j$$

for some $0 < \theta < 1$. The third term on right above is non-negative since G is convex. Hence $G(x+y) \geqq G(x) + \langle \text{grad } G(x), y\rangle$. Let $u = e^x$, $v = e^y$. Then $\text{grad}\, G(x) = (1/F(u))\, u\, \text{grad}\, F(u)$, and the statement of the lemma is immediate.

If in addition to the hypothesis above we have that $F(u)$ and $u\, \text{grad}\, F(u)$ extend to be continuous on R^n_0, then the inequality $F(uv)/F(u) \geqq v^{u\, grad\, F(u)/F(u)}$ continues to hold for $v \in R^n_+$ and any $u \in R^n_0$ for which $F(u) \neq 0$.

To prove the theorem, we will apply the lemma to the function $Q(y)$. That $\log Q(e^y)$ is convex follows from Lemma 3. The remaining extended hypotheses are obviously true. Let $x \in F(r+1, r+2, \cdots, n)$, a regularly defined flat. Let $\alpha \geqq 0$, $\beta > 0$, $\alpha + \beta = 1$. Put $u = A(x)$, and let $v \in R^n_+$ be such that $A(x)v = \alpha A(x) + \beta A(h(x))$. That such a v exists is clear since $A(x)$ and $A(h(x))$ have the same support. Then the extended Lemma 10 yields that

$$\frac{P(\alpha x+\beta h(x))}{P(x)}=\frac{Q(A(\alpha x+\beta h(x)))}{Q(A(x))}\geqq v^{A(h(x))}$$

$$=\prod_{i=1}^{r} v_i^{Ai(h(x))}=\prod_{i=1}^{r}\left\{\frac{\alpha A_i(x)+\beta A_i(h(x))}{A_i(x)}\right\}^{A_i(h(x))}$$

$$\geqq\prod_{i=1}^{r}\left\{\frac{A_i(x)^{\alpha}A_i(h(x))^{\beta}}{A_i(x)}\right\}^{A_i(h(x))}=\prod_{i=1}^{r}\left\{\frac{A_i(h(x))}{A_i(x)}\right\}^{\beta A_i(h(x))}$$

$$=\left\{\prod_{i=1}^{r}\left[\frac{\frac{1}{d}A_i(h(x))}{\frac{1}{d}A_i(x)}\right]^{A_i(h(x))/\alpha}\right\}^{\beta d}.$$

Now if $p, q \in R_0^r$ satisfy $\Sigma p_i = \Sigma q_i = 1$, then as a function of p, p^q achieves its maximum uniquely at $p = q$. Hence the quantity enclosed in braces above is greater than or equal to one, and equals one if and only if $A_i(h(x) = A_i(x)$, $1 \leqq i \leqq r$, which implies $h(x) = x$. The theorem is the special case $\alpha = 0$, $\beta = 1$

It is worth pausing now to interpret the above results for the permanent. In this case we take $A = R^{k^2}$, points of A will be doubly indexed sequences $y_{i,j}$, $1 \leqq i$, $j \leqq k$, and for K the set of doubly stochastic matrices D_k. $\bar{K}$ is the set of generalized doubly stochastic matrices (entries arbitrary in sign). For S we simply take co-ordinate functions themselves, so that $A(x) = x$. $\mathscr{E}$ is the set of permutation matrices, and so $Q(y) = \Sigma_{\pi\in\mathscr{E}}\, y^{\pi}$, and $P(x) = \Sigma_{\pi\in\mathscr{E}}\, x^{\pi}$ is just the permanent of x. Then $h(x) = (1/P(x))\,\Sigma x^{\pi}\pi$. Since h is a bijection, we have the statement that every doubly stochastic matrix y may be written in the form

$$y=\frac{1}{P(x)}\Sigma x^{\pi}\pi$$

for a unique choice of doubly stochastic x.

Consider the set of all representatives $y = \Sigma p_{\pi}\pi$ of y as a convex sum of permutation matrices. I believe the representation above is the unique one for which $\Pi_{\pi}\, p_{\pi}^{p_{\pi}}$ is minimized.

We also have the statement that $P(l(x)) \leqq P(x)$, $x \in K$, with equality if and only if $x = l(x) = h(x)$, but the map l has a complicated nature, even though its inverse h is relatively simple. However, for the case at hand, a somewhat simpler map which does not increase the permanent can be found.

LEMMA 11. *Let $x \in D_k$, x not a permutation matrix. Then*

$$P\left(\frac{x-P(x)h(x)}{1-P(x)}\right)\leqq P(x)$$

with equality if and only if $h(x) = x$.

We are going to apply Lemma 10 to the function $F(y) = \Pi_{i=1}^{k}(\Sigma_{j=1}^{k} y_{ij}) - Q(y)$. $F(y)$ is a homogeneous polynomial with non-negative coefficients, and satisfies the extended hypothesis of Lemma 10. $\log F(e^y)$ is convex by Lemma 3, since it is the Laplace transform of a measure. We select $u = x$, x not a permutation matrix, so that $F(u) \neq 0$. We find $u \operatorname{grad} F(u)/F(u)$ equal to

$$\frac{x - P(x)h(x)}{1 - P(x)}.$$

We pick $v \in R_+^{k^2}$ such that $xv = (x - P(x)h(x))/(1 - P(x))$, and we obtain

$$\frac{F(xv)}{F(x)} = \frac{1 - P\left(\dfrac{x - P(x)h(x)}{1 - P(x)}\right)}{1 - P(x)} \geqq v^{(x - P(x)h(x))/(1 - P(x))}.$$

Repeating the last argument used in the proof of Lemma 10, we easily find that the right-hand side is greater than or equal to one, with equality only for $h(x) = x$.

We return now to the general case, and want to investigate when, for y and $y' \in R_0^n$, we have that $q(y) = q(y')$. Let $F = F(r+1, r+2, \cdots, n)$ be a regularly defined flat and suppose that both y and y' have F-like support. Then according to Lemma 5, there exist $\beta_1, \beta_2, \cdots, \beta_r > 0$ such that $y_i = \beta_i y_i'$ and such that $\Pi_{i=1}^{r} \beta_i^{A_i(x)} = \alpha$ (constant) for all $x \in F$. Since we are in the homogeneous case, we can equally well assert that there exist $\gamma_1, \gamma_2, \cdots, \gamma_r > 0$ and a positive constant δ such that $y_i = \delta\gamma_i y_i'$ and such that $\Pi_{i=1}^{r} \gamma_i^{A_i(x)} = 1$ for all $x \in F$. The last is equivalent to asserting that $\Pi_{i=1}^{r} \gamma_i^{A_i(e)} = 1$ for all $e \in \mathscr{E} \cap F$.

We introduce the semigroup $S = \{\gamma \in R_+^n \mid \gamma^{A(x)} \geqq 1 \ \forall x \in K\}$., and the group $G = \{\gamma \in s \mid \gamma^{A(x)} = 1 \,\forall\, x \in k\}$.

LEMMA 12. *S is a closed convex set and G is the set of its extreme points.* For let $\gamma, \gamma' \in S$, $p, q \geqq 0$, $p + q = 1$. Then

$$(p\gamma + q\gamma')^{A(x)} = \prod_{i=1}^{n} (p\gamma_i + q\gamma_i')^{A_i(x)} \geqq \prod_{i=1}^{n} (\gamma_i^p \gamma_i'^q)^{A_i(x)} \geqq 1.$$

This proves S is convex.

Let $x \in \underline{K}$, so that $A_i(x) > 0$, $i = 1, 2, \cdots, n$. Then $\gamma^{A(x)} = \Pi_{i=1}^{n} \gamma_i^{A_i(x)} \geqq 1$. Now if γ^ν is a sequence of elements in S, and any one of the coordinates of this sequence is approaching zero, then one of the remaining coordinates must be approaching infinity. This proves S is closed.

To see that every point of G is an extreme point, suppose $g = pg' + qg''$, $g, g', g'' \in G$, $p, q \geqq 0$, $p + q = 1$, and $x \in \underline{K}$. Then

$$(pg' + qg'')^{A(x)} = \prod_{i=1}^{n} (pg'_i + qg''_i)^{A_i(x)} \geqq \left[\prod_{i=1}^{n}(g'_i)^{A_i(x)}\right]^p \left[\prod_{i=1}^{n} (g''_i)^{A_i(x)}\right]$$

with equality holding only if $g' = g''$, so points of G are extreme points.

Next let $\gamma \in S$. Define $\mathscr{E}_1 = \{e \in \mathscr{E} \mid \gamma^{A(e)} = 1\}$. Suppose, for the sake of argument, that $A_1(e) = 0$ for every $e \in \mathscr{E}_1$. Then we can make γ_1 smaller and still maintain $\gamma^{A(e)} \geqq 1$ for all $e \in \mathscr{E}$. So we replace γ_1 by a suitable smaller γ'_1. Put $\gamma' = (\gamma'_1, \gamma_2, \cdots, \gamma_n)$ and $\gamma'' = (2\gamma_1 - \gamma'_1, \gamma_2, \cdots, \gamma_n)$. Then $\gamma = \gamma'/2 + \gamma''/2$, $\gamma' \in S$, and $\gamma'' \in S$ since the coordinates of γ'' are at least as great as those of γ'. Finally γ' and γ'' are distinct.

So if γ is to be an extreme point, then for every $1 \leqq i \leqq n$, $A_i(e) \neq 0$ for some $e \in \mathscr{E}_1$. Let the cardinality of $\mathscr{E}_1$ be ρ and put $x = (1/\rho)\sum_{e \in \mathscr{E}_1} e$. Then $x \in \underline{K}$, since $A_i(x) > 0$ $\forall i$, so x may be written $x = \sum_{e \in \mathscr{E}'} p_e e$ as a proper convex combination of all the extreme points of $\underline{K}$. But then

$$1 = \gamma^{A(x)} = \gamma^{\Sigma p_e A(e)} = \prod_{e \in \mathscr{E}'} (\gamma^{A(e)})^{p_e},$$

and the right-hand side, which is greater than or equal to one, can only be equal to one if $\gamma^{A(e)} = 1$ $\forall e \in \mathscr{E}'$, and so for $\forall$ $e \in \mathscr{E}$; that is, $\gamma \in G$.

Note that we are not asserting that S is the convex hull of its extreme points.

COROLLARY 13 (to the proof). *If $\gamma \in S$, $\exists \gamma' \in G$ with each coordinate of γ' no greater than the corresponding one of γ.*

LEMMA 14. *Suppose $F = F(r+1, r+2, \cdots, n)$ is a regularly defined flat and that $\beta_1, \beta_2, \cdots, \beta_r$ are positive numbers such that $\Pi_{i=1}^{r} \beta_i^{A_i(x)} = 1$ for all $x \in F$. Then there exists $\gamma \in G$ such that $\beta_i = \gamma_i$ for $i = 1, 2, \cdots, r$.*

For the proof it will suffice to suppose that F is a face of K, for our argument is such that it enables us to advance from any flat to a second flat of which the first is a face.

Let the dimension of K be ρ, so the dimension of F is $\rho - 1$. Pick in F any ρ extreme points $e_1, e_2, \cdots, e_\rho$ such that $\bar{F}$ is the affine subspace determined by these. Let $e_{\rho+1}$ be an extreme point of K not in $\bar{F}$, so that $\bar{K}$ is determined by $e_1, e_2, \cdots, e_{\rho+1}$. Select positive numbers $\beta_{r+1}, \beta_{r+2}, \cdots, \beta_n$ such that $\Pi_{i=1}^{n} \beta_i^{A_i(e_{\rho+1})} = 1$. Now any

$x \in K$ may be written as a strictly improper convex combination (coefficients arbitrary in sign but of coefficient sum one) of $e_1, e_2, \cdots, e_{\rho+1}$; that is, $x = \sum_{v=1}^{\rho+1} p_v e_{vy}$. But then

$$\prod_{i=1}^{n} \beta_i^{A_i(x)} = \prod_{i=1}^{n} \prod_{v=1}^{\rho+1} \beta_i^{p_v A_i(e_v)} = 1.$$

COROLLARY 15. (to Lemma 5). *If y and y' have F-like support for some flat F, and $q(y) = q(y')$, then there exists $\gamma \in G$ and positive constant δ such that $y_i = \delta\gamma_i y_i'$, $i = 1, 2, \cdots, n$.*

δ is unique if $F = F(r+1, r+2, \cdots, n)$ is regularly defined, the numbers $\gamma_1, \gamma_2, \cdots, \gamma_r$ are also unique.

THEOREM 16. *Let $F = F(r+1, r+2, \cdots, n)$ be regularly defined and y have F-like support. Then there exists an $x \in K$, a $\delta > 0$ and a $\gamma \in G$ such that $y = \delta\gamma A(x)$. δ is unique; $x \in \underline{F}$ and is unique.*

Consider $z = A(l(q(y))) \in R_0^n$; z clearly has F-like support. But $q(z) = h(l(q(y))) = q(y)$. Put $x = l(q(y))$, $x \in F$. By Corollary 15, $y = \delta\gamma A(x)$, and δ is unique. If also $y = \delta\gamma' A(x')$, then $q(y) = h(x) = h(x')$, implying $x = x'$.

For the case of doubly stochastic matrices, the above is essentially the so-called D_1AD_2 theorem [1], [6], save for the description of the group G, which is obtained later in this paper. It remains to inquire into the uniqueness of γ. To this end, we say $x \in K$ is indecomposable if for each $1 \leqq i \leqq n$, there exists $e \in \mathscr{E}$ for which $A_i(e) \neq 0$ and such that $\Pi_{v \neq i} A_v(x)^{A_v(e)} \neq 0$. With F-like support, y is called indecomposable if some (hence any) $x \in \underline{F}$ is indecomposable.

THEOREM 17. *Let y in Theorem 16 be indecomposable. Then γ is unique.*

For suppose $y = \delta\gamma A(x) = \delta\gamma' A(x)$. Put $\beta = \gamma\gamma'^{-1}$. Then $\beta A(x) = A(x)$; that is, $\beta_v A_v(x) = A_v(x)$. Then

$$\prod_{\substack{v=1 \\ v\neq i}}^{n} \beta_v^{A_v(e)} \prod_{\substack{v=1 \\ v\neq i}}^{n} A_v(x)^{A_v(e)} = \prod_{\substack{v=1 \\ v\neq i}}^{n} A_v(x)^{A_v(e)}.$$

Since $\Pi_{v\neq i} \beta_v^{A_v(e)} = 1/\beta_i^{Ai(e)}$, and we may select e such that $A_i(e) \neq 0$ and such that $\Pi_{v\neq i} A_v(x)^{A_v(e)} \neq 0$, we have $\beta_i = 1$, for $i = 1, 2, \cdots, n$.

For the case of doubly stochastic matrices, what we have called indecomposability above corresponds to what is called there complete indecomposability. The equivalence of the two follows readily from König's theorem.

We want to obtain a characterization of the number δ appearing in Theorem 16. To this end we define for $y \in R_0^n$,

$$E(y) = \frac{1}{d} \min_{s \in S} \sum_{i=1}^{n} y_i s_i .$$

By virtue of Corollary 13, it is obvious that

$$E(y) = \frac{1}{d} \min_{\gamma \in G} \sum y_i \gamma_i .$$

THEOREM 18. *Let* $y = \delta \rho A(x)$, $\delta > 0$, $\rho \in G$, $x \in K$. *Then*

$$E(y) = \delta = \left[\frac{Q(y)}{P(l(q(y)))} \right]^{1/d} .$$

We have $1/d \min_{\gamma \in G} \sum y_i \gamma_i = \delta/d \min_{\gamma \in G} \sum A_i(x)\gamma_i \geqq \delta \prod_{i=1}^{n} \gamma^{(1/d)A_i(x)} = \delta$, by the inequality of the arithmetic and geometric mean. On the other hand, $1/d \sum y_i \rho_i^{-1} = \delta$.

But also, if $y = \delta \rho A(x)$, we have seen in Theorem 16 that $x = l(q(y))$. Note as well that $Q(y) = \delta^d Q(A(x)) = \delta^d P(x)$, so we obtain the remaining equality o the theorem.

THEOREM 19. *$E(y)$ is continuous. If $Q(y) = 0$, then $E(y) = 0$. Otherwise*

$$E(y) = \left[\frac{Q(y)}{P(l(q(y)))} \right]^{1/d} .$$

First let us note that the function described above is continuous. For if $Q(y) \neq 0$, $P(l(q(y)))$ is uniformly bounded away from zero, since P is bounded away from zero on K.

Now let y^v be a sequence approaching y, and let γ be arbitrary in G. Since $1/d \sum y_i^v \gamma_i \geqq E(y^v)$, we have $1/d \sum y_i \gamma_i \geqq \limsup_{v \to \infty} E(y^v)$, so $E(y) \geqq \limsup E(y^v)$.

Then let y^v be a sequence such that $y_i^v > 0$ for all i and v, and such that y_i^v decreases monotonically to y_i. Clearly, then, $E(y^v) \geqq E(y)$, so $\limsup E(y^v) \geqq E(y)$. Since $E(y^v)$ is monotone, $\lim E(y^v)$ exists. Hence $E(y) = \lim E(y^v)$, for any positive sequence converging monotonically to y. But then, since y^v has K-like support, we know that

$$E(y^v) = \left[\frac{Q(y^v)}{P(l(q(y^v)))} \right]^{1/d} ,$$

and passage to limit gives desired result.

There are several more properties of E worth singling out, but first we need an intermediate result.

LEMMA 20. *Suppose* $y \in R_0^n$, *and* $y_1, y_2, \cdots, y_r \neq 0$, $y_{r+1}, y_{r+2}, \cdots, y_n = 0$. *Put* $F = F(r+1, r+2, \cdots, n)$. *If* F *is empty, which is so if and only if* $Q(y) = 0$, *put* $\tilde{y} = 0$. *Otherwise, let* $F' = F(s+1, s+2, \cdots, n)$, $s \leqq r$, *be the same flat regularly defined, and put* $\tilde{y}_i = y_i$ *for* $i = 1, 2, \cdots, s$, *and* $\tilde{y}_i = 0$ *for* $i \geqq s+1$. *Then* $Q(y) = Q(\tilde{y})$. *If* $Q(y) \neq 0$, *then* $q(y) = q(\tilde{y})$. *Hence always* $E(y) = E(\tilde{y})$.

The result is obvious by inspection.

THEOREM 21.

i. E *is homogeneous of degree one, and* $E(x+y) \geqq E(x) + E(y)$;

ii. $E(\gamma x) = E(x)$ *for* $\gamma \in G$;

iii. $E(A(x)) \equiv 1$ *for* $x \in K$;

iv. $E(x^p y^q) \leqq E^p(x) E^q(y)$, $p + q = 1$, $p, q \geqq 0$;

v) $E(y) = \max_{x \in K} \left[\dfrac{y^{A(x)}}{A(x)^{A(x)}} \right]^{1/d}$.

(i) and (ii) are obvious from the definition of $E(y)$, (iii) from Theorem 18. As for (iv), let γ and ρ be elements of G such that $1/d \sum x_i \gamma_i$ and $1/d \sum y_i \rho_i$ are within ε of $E(x)$ and $E(y)$ respectively. Then since $\gamma^p \rho^q$ is also an element of G,

$$E(x^p y^q) \leqq \frac{1}{d} \sum x_i^p y_i^q \gamma_i^p \rho_i^q \leqq \left(\frac{1}{d} \sum x_i \gamma_i \right)^p \left(\frac{1}{d} \sum y_i \rho_i \right)^q$$

which gives the desired result.

For the proof of (v) put

$$F(y) = \max_{x \in K} \left[\frac{y^{A(x)}}{A(x)^{A(x)}} \right]^{1/d}.$$

and note first that if $Q(y) = 0$, then $y^{A(e)} = 0 \ \forall\ e \in \varepsilon \in \mathscr{E}$, hence $y^{A(x)} = 0 \ \forall\ x \in K$, so that $F(y) = 0$. Equally obvious is the assertion that $F(y) = 0$ implies $Q(y) = 0$. Now let y be as described in Lemma 20, and suppose F is not empty. In seeking the maximum of $y^{A(x)}/A(x)^{A(x)}$, it is clear that we can confine x to F'. Thus $F(y) = F(\tilde{y})$. But since $\tilde{y}$ has F'-like support we can write $\tilde{y} = \delta \gamma A(u)$ for some $u \in \underline{F}'$. But then

$$\frac{\tilde{y}^{A(x)}}{A(x)^{A(x)}} = \delta^d \frac{A(u)^{A(x)}}{A(x)^{A(x)}},$$

and as we have argued before, $A(u)^{A(x)}/A(x)^{A(x)}$ achieves its maximum uniquely at $x = u$. Hence $F(\tilde{y}) = \delta$, but by Lemma 20, $E(\tilde{y}) = E(y)$.

LEMMA 22. *Let $y \in R_0^n$, $y \neq 0$, and suppose $\langle y, \alpha \rangle = \sum_{i=1}^r y_i \alpha_i$, $\alpha \in G$, has a critical point. Then this critical point is a minimum, and y has F-like support for some flat F. If y is indecomposable, the critical point is unique.*

Let $\alpha \in G$ be a critical point, let γ be arbitrary in G, and put $z = y\alpha$. Then $\langle z, \gamma^t \rangle$ has a vanishing derivative at $t = 0$. Thus

$$\frac{d}{dt}\langle z, \gamma^t \rangle (t = 0) = \sum_{i=1}^n z_i \log \gamma_i = 0.$$

Since $\log \gamma$ runs precisely through the orthogonal complement of the set of points $A(e)$, $e \in \mathscr{E}$, we must have $z = \sum_{e \in \mathscr{E}} \eta_e A(e)$. Let $\eta = \sum \eta_e$. If $\eta \neq 0$, we may write $z = \eta A(x)$ for some $x \in \bar{K}$. If $\eta > 0$, then since $z \in R_0^n$, $x \in K$, and z and y have same support as $A(x)$.

Suppose $\eta < 0$. Then $A_i(x) \leqq 0$, $1 \leqq i \leqq n$. Let $u \in K$, $t \geqq 0$. Then $A_i((1 + t)u - tx) = (1 + t)A_i(u) - tA_i(x) \geqq 0$, implying $u + t(u - x) \in K$ for all $t \geqq 0$. Since K is compact we must have $u = x$, whence $A_i(x) = 0$, so $z = y = 0$, contradicting our original hypothesis.

Suppose $\eta = 0$. $z = \sum \eta_e A(e) = \sum \eta_e A(e) + A(0) - A(0) = A(x) - A(0)$ for some $x \in A$. And $A_i(x) - A_i(0) \geqq 0$. Again let $u \in K$. Then $A_i(u + tx) = A_i(u) + t(A_i(x) - A_i(0)) \geqq 0$, implying $u + tx \in K$ for all t, which implies $x = 0$. Hence $z = A(x) - A(0) = 0 = y$, again contradicting our original hypothesis.

So we must have $z = \eta A(x)$, $x \in K$, $\eta > 0$, and so $y = \eta \alpha^{-1} A(x)$. By Theorem 18, we know that $\min_{\gamma \in G} \langle y, \gamma \rangle = d\eta$. But also $\langle y, \alpha \rangle = d\eta$, showing the critical point is a minimum. The uniqueness of the critical point in case y is indecomposable follows from Theorem 17.

THEOREM 23. *Let $T = \{ y \in R_0^n \mid y_1, y_2, \cdots, y_r > 0; \quad y_{r+1}, y_{r+2}, \cdots, y_n = 0 \}$. Then E restricted to T is a real analytic function. Q never vanishes or vanishes identically on T. If $Q \not\equiv 0$, define for each $y \in T$, F, F' and $\tilde{y}$ as in Lemma 20, and write $\tilde{y} = \delta \gamma A(x)$. Then $y_j(\partial E / \partial y_j) = 0$ for $j = s + 1, \cdots, r$; and $y_j(\partial E / \partial y_j) = (\delta / d) A_j(x)$ for $j = 1, 2, \cdots, s$. In case $Q \equiv 0$ on T, then $E \equiv 0$ on T, and so all derivatives exist and are zero.*

The final assertion of the theorem is obvious, as is the statement about the vanishing of Q on T. So we confine our attention to the case $Q \not\equiv 0$ on T. We know then that $E(y)^d = Q(y)/P(l(q(y)))$. As y ranges over T, $q(y)$ which is real analytic ranges over points in K all belonging to the interior of the same flat. Hence $l(q(y))$ is real analytic. Since $P(l(q(y)))$ is uniformly bounded away from zero, and $Q(y)$ is different from zero, $E(y)$ is real analytic.

It is possible now to compute the derivatives of E directly from the definition, but it is easier to proceed as follows. By virtue of Theorem 19 and Lemma 20, it is clear that E does not depend on the $(s+1)$st to the rth coordinate, $\partial E/\partial y_j = 0$ for $j = s+1, s+2, \cdots, r$. Also $(\partial E/(\partial y_j)(y) = (\partial E/\partial y_j)(\tilde{y})$ for $j = 1, 2, \cdots, s$.

For notational convenience, we denote $\partial E/\partial y_j$ by E_j, and we extend the definition of $E_j(y)$ by setting it equal to zero for $j = s+1, s+2, \cdots, n$. Since E is homogeneous of degree one, $\sum_{j=1}^n y_j E_j(y) = E(y)$. Since for any $\rho \in G$, $E(\rho y) = E(y)$, we have $\rho_i E_i(\rho y) = E_i(y)$. Also for any scalar t, $E(\rho^t y) = E(y)$, from which we obtain, on differentiating with respect to t and setting $t = 0$, that $\sum_{i=1}^n y_i E_i(y) \log \rho_i = 0$. As in the proof of Lemma 22, the above implies that $y_i E_i(y) = \sum_e \eta_e A_i(e)$. Summing on i, we obtain $E(y) = d\eta$, $\eta = \sum_e \eta_e$, and $\eta > 0$. Since $E(y)$ is a monotone non-decreasing function of each argument, $y_i E_i(y) \geqq 0$. Referring again to the proof of Lemma 22, we obtain that $yE(y) = \eta A(x)$ for some $x \in K$, indeed for some $x \in F'$.

Now let $v \in \underline{F}'$, w be any point in $\bar{F}'$, and t be small. Then $E(A(v + t(v-w))) \equiv 1$. Differentiating with respect to t and then setting $t = 0$, we obtain $\sum_{i=1}^n E_i(A(v))[A_i(w) - A_i(v)] = 0$ or $\sum_{i=1}^n E_i(A(v))A_i(w) = E(A(v)) = 1$.

We know from just above that $A_i(v)E_i(A(v)) = (1/d)A_i(z)$ for some $z \in F'$. Hence $\sum_{i=1}^s (A_i(z)/A_i(v))A_i(w) = d$ for any $w \in \bar{F}'$ from which we obtain $\sum_{i=1}^s [(A_i(z) - A_i(v))/A_i(v)]A_i(w) = 0$ which gives $\sum_{i=1}^s [(A_i(z) - A_i(v))^2/A_i(v)] = 0$, implying $v = z$. Thus $E_i(A(v)) = 1/d$ for $i = 1, 2, \cdots, s$.

Now write $\tilde{y} = \delta\gamma A(x)$. Since E_i is homogeneous of degree 0, and since $\rho_i E_i(\rho y) = E_i(y)$ for any $\rho \in G$, we obtain

$$\gamma_i^{-1} E_i(\delta A(x)) = \frac{1}{d}\gamma_i^{-1} = E_i(\tilde{y}) = E_i(y)$$

for $1 \leqq i \leqq s$, which completes the proof.

It is worth remarking now that if we know E in the neighborhood of a point y, we can compute immediately the decomposition of $\tilde{y}$ in form $\delta\gamma A(x)$.

It is also worth interpreting the above for the space D_k of $k \times k$ doubly stochastic matrices sitting in the affine space R^{k^2}. One finds readily that the group G consists of matrices whose i, jth entry is $\lambda_i u_j$ subject to $\Pi_i \lambda_i = \Pi_j u_j = 1$. Hence $E(y) = (1/k)\min_{\lambda,u} \sum y_{ij}\lambda_i u_j$ subject to above constraints.

It is possible to obtain a simple direct proof of the D_1AD_2 theorem just by considering the critical points of the above variational problem.

THEOREM 24. *Let M and m be upper and lower bounds for $P(x)$. Then for any $y \in R_0^n$, $ME^d(y) \geqq Q(y) \geqq mE^d(y)$.*

If $Q(y) = 0$, the statement is clear. Otherwise let y and $\tilde{y}$ be as in Lemma 20, and write $\tilde{y} = \delta\gamma A(x)$. Then $Q(y) = Q(\tilde{y}) = \delta^d Q(A(x)) = E^d(y)P(x)$, and we are done.

If $Q(y)$ is homogeneous of degree $d \leqq 1$, then it is easy to see that $Q(y)$ is a concave function, since each of the summands is, so the minimum of $P(x)$ is attained at an extreme point of K. This simple observation leads to our next result.

THEOREM 25. *Let $\lambda = \min_{e \in \mathscr{E}} c(e)A(e)^{(1/d)A(e)}$, and assume $d > 1$. Then for all $x \in K$, $P(x) \geqq \lambda(d/n)^{d-1}$.*

First we note for any $y \in K$ that by our remarks above $Q(A^{1/d}(y))$ is a concave function on K, so $QA(^{1/d}(y)) \geqq \lambda$.

Next, let x be arbitrary in K, and write $A^d(x) = \delta\gamma A(y)$, $\delta > 0$, $\gamma \in G$, $y \in K$, so that $A(x) = \delta^{1/d}\gamma^{1/d}A^{1/d}(y)$. Also $\delta = E(A^d(x))$, and $P(x) = Q(A(x)) = \delta Q(A^{1/d}(y)) \geqq \lambda E(A^d(x))$. Let I be the point in R_0^n with all coordinates equal to one. Then by Theorem 21,

$$1 = E(A(x)) = E(A(x)I) \leqq E^{1/d}(A^d(x)) \cdot E^{(d-1)/d}(I),$$

so $E(A^d(x)) \geqq 1/E^{d-1}(I)$. But $E(I) = (1/d)\min_{\gamma \in G} \sum \gamma_i \leqq n/d$. Hence $P(x) \geqq \lambda(d/n)^{d-1}$.

COROLLARY 26. *If $x \in D_k$, then* $\mathrm{Perm}(x) \geqq 1/k^{k-1}$.

COROLLARY 27. *Suppose it is known that $A(x)$ has at least r ($r < n$) components zero. Then $P(x) \geqq \lambda(d/(n-r))^{d-1}$.*

For the proof, replace I in the argument above by the point I' which has zero components matching those of $A(x)$, components one elsewhere. Then $E(I') \leqq (n-r)/d$, and the proof is completed as before.

A simple combinatorial argument which we now present gives a result in some respects better than Corollary 26. Fix an integer $0 < p \leqq n$ and let V be the collection of points in R_0^n which have p coordinates equal to one, the remainder being zero.

LEMMA 28. *Let $x = (x_1, x_2, \cdots, x_n) \in R_0^n$ and let $\eta = \sum_{i=1}^n x_i$. In order for x to be written as a linear combination with non-negative coefficients of elements of V, it is necessary and sufficient that for all $1 \leqq v \leqq n$, $px_v \leqq \eta$.*

The necessity of the condition is obvious. The proof of the sufficiency is by induction on $s = n - p$. Let us first note that the condition is sufficient in case $s = 0$, since the collection of inequalities $nx_v \leqq \eta$ implies $n\eta = \sum_i nx_i \leqq n\eta$, and since we have equality, we must always have $nx_v = \eta$, that is, all coordinates of x are equal.

Let x be a point in R_0^n and $p < n$. If any coordinate of x is zero, we can discard it, and we are reduced to a value of s less by one, and the result is true. Suppose that $r, 0 \leqq r \leqq p$, of the inequalities $px_v \leqq \eta$ are equalities, say $px_1 = \eta$, $px_2 = \eta, \cdots, px_r = \eta$. If $r = p$, then $\eta \geqq \sum_{i=1}^p x_i = \eta$, and since we have equality, it must be that $x_{p+1} = x_{p+2} = \cdots = x_n = 0$, and we are again reduced to an earlier case, so the result is true.

We now describe a procedure which reduces a given case to one where some component of x is zero, or increases by one the number of inequalities which are equalities, which enables us to complete the inductive proof.

So we suppose no coordinate of x is zero, that $px_1 = \eta$, $px_2 = \eta, \cdots, px_r = \eta$, $0 \leqq r < p$, and $px_v < \eta$, $v = r+1, r+2, \cdots, n$. Put $\alpha = \min[x_1, x_2, \cdots, x_p, (\eta - px_{p+1})/p, (\eta - px_{p+2})/p, \cdots, (\eta - px_n)/p]$ and $y = (x_1 - \alpha, x_2 - \alpha, \cdots, x_p - \alpha, x_{p+1}, x_{p+2}, \cdots, x_n)$. Then $y \in R_0^n$ and $x = \alpha v + y$ for $v \in V$. The sum of the coordinates of y is $\eta - p\alpha$, and it is easy to verify that $py_v \leqq \eta - p\alpha$ for all v, and $py_i = \eta - p\alpha$ for $i = 1, 2, \cdots, r$. If $\alpha = x_i$ for some $1 \leqq i \leqq p$, then $y_i = 0$. If $\alpha = (\eta - px_i)/p$ for some $p + 1 \leqq i \leqq n$ then $py_i = \eta - p\alpha$. So either y has a zero coordinate or $r + 1$ of the inequalities for y are equalities.

THEOREM 29. *Let $x \in D_k$ and let r be an integer $\leqq k^{k/(k+1)}$. Then the sum of the r largest terms in the expression for* Perm(x) *is $\geqq r/k^k$, with equality if and only if x is the doubly stochastic matrix f having all entries equal.*

First note that x^x has a minimum of $1/k^k$ achieved uniquely at $x = f$. Let $x = \sum_{v=1}^{k!} p_v \pi_v$. Suppose for all $1 \leqq i \leqq k!$ that $rp_i \leqq 1$. Let p be the vector with ith coordinate p_i. By the last lemma, we may write $p = (1/r)\sum_v q_v h_v$, where each h_v is a vector with r entries equal to one, remaining entries zero, v being an integer in range $1 \leqq v \leqq \binom{k!}{r}$. Also $q_v \geqq 0$ and $\sum q_v = 1$. Putting back in the expression for x the expressions so obtained for the p_i, we achieve a representation $x = (1/r)\sum q_v\theta_v$, where each θ_v is a sum of r distinct permutation matrices.

Then $x^x = \Pi_v x^{\theta_v q_v/r} \geqq 1/k^k$. If we always have $x^{\theta_v/r} \leqq 1/k^k$, then we must have $x^{\theta_v/r} = 1/k^k$, so $x^x = 1/k^k$, and $x = f$. Otherwise, we have, say, $x^{\theta_1/r} > 1/k^k$, and $\theta_1 = \pi_1 + \pi_2 + \cdots + \pi_r$, from which it follows by the inequality of arithmetic and geometric mean that $(1/r)\sum_{i=1}^r x^{\pi_i} > 1/k^k$.

The remaining alternative is that for some i, $rp_i > 1$. But then already $x^{\pi_i} > 1/r^k \geqq r/k^k$, completing the proof.

There is an interesting complement to the last result. Let I be a set of k permutation matrices, each of degree k. We say they are independent if the square matrix $\sum_{\pi \in I}\pi$ has every entry one.

THEOREM 30. *Let I be an independent set and $x \in D_k$. Then if $x \neq f$, $x^\pi < 1/k_k$ for some $\pi \in I$.*

We may write $x = \sum_{\pi \in I}\lambda(\pi)\circ\pi$, where $\lambda(\pi)$ are diagonal matrices, and $\lambda(\pi)\circ\pi$ is matrix multiplication. It is easy to see that $E(\lambda(\pi)\circ\pi) = [\Pi_{i=1}^k \lambda_i(\pi)]^{1/k} = x^{\pi/k}$. We have $1 = E(x) \geqq \sum_{\pi \in I}E(\lambda(\pi)\circ\pi) \geqq \sum_{\pi \in I}x^{\pi/k}$. If for some $\pi \in I$, $x^{\pi/k} > 1/k$, then for some other $\pi' \in I$, $x^{\pi'/k} < 1/k$. Otherwise we have for all $\pi \in I$, $x^{\pi/k} = 1/k$. But then $1 = E(x) = (1/k)\sum_{i,j}x_{ij} = (1/k)\sum_{i,j}\sum_{\pi \in I}\lambda_i(\pi)\pi_{ij} \geqq \sum_{\pi \in I}E(\lambda(\pi)\circ\pi) = 1$. So we must have for any $\pi \in I$ that $(1/k)\sum_i^k\lambda_i(\pi) = [\Pi_{i=1}^k\lambda_i(\pi)]^{1/k}$, implying that $\lambda_i(\pi) = 1/k$, implying that $x = f$.

We return once more to the general case, and want to give a duality result, which can possibly be used to improve Theorem 25. For any $e \in \mathscr{E}$, we define the semigroup $S(e) = \{y \in R_0^n | y^{A(e)} \geqq 1\}$ and the semigroup $G(e) = \{y \in R_0^n | y^{A(e)} = 1\}$ If $g \in G$, then $g\,G(e) = G(e)$. Note that $S(e)$ is a convex set, and that if $y \in R_0^n$, then $\min_{s \in G(e)}\langle y, s\rangle = d(y^{A(e)/d}/A(e)^{A(e)/d})$. Let $\mathscr{L}'$ be the convex set consisting of all $l \in R_0^n$ which may be written $l = \sum_{e \in \mathscr{E}}c(e)A(e)^{A(e)/d}s(e)$ with $s(e) \in S(e)$, and $\mathscr{L}$ the set which may be written as above but with $s(e) \in G(e)$. Note that if $l \in \mathscr{L}'$, then $l \in R_+^n$.

THEOREM 31. *Let $m = \min_{x \in K}P(x)$, and suppose $d > 1$. Then*

$$\frac{1}{m} = \max_{l \in \mathscr{L}} E^{d-1}\left(l^{-\frac{1}{(d-1)}}\right) = \max_{l \in \mathscr{L}} Q^{d-1}\left(l^{-\frac{1}{(d-1)}}\right).$$

Also

$$m = \min_{x \in K, l \in \mathscr{L}} l^{A(x)/d}A(x)^{(d-1)A(x)/d}.$$

From our remarks above, it is clear that $Q(y) = (1/d)\min_{l \in \mathscr{L}}\langle l, y^d\rangle$. But also then, $Q(y) = (1/d)\min_{l \in \mathscr{L}}\min_{g \in G_i}\langle lg, y^d\rangle = \min_{l \in \mathscr{L}}E(ly^d)$. Since $d > 1$,

$$E(y) = E(l^{-1/d}l^{1/d}y) \leqq E^{d-1/d}(l^{-1/(d-1)}) \cdot E^{1/d}(ly^d).$$

Since $E(A(x)) \equiv 1$ for $x \in K$, we obtain

$$E(lA^d(x)) \geqq \frac{1}{E^{d-1}(l^{-1(d-1)})}\,.$$

Thus

$$P(x) = Q(A(x)) \geqq \frac{1}{\max_{l \in \mathscr{L}} E^{d-1}(l^{-1/(d-1)})}$$

and so $1/m \leqq \max_{l \in \mathscr{L}} E^{d-1}(l^{-1/(d-1)})$.

But from Theorem 24, we have $E^d(l^{-1/(d-1)}) \leqq (1/m)Q(l^{-1/(d-1)})$, and this, in conjunction with the last inequality, yields

$$\frac{1}{m} \leqq \max_{l \in \mathscr{L}} Q^{d-1}(l^{-1/(d-1)}).$$

Repeating the argument used above, $Q(l^{-1/(d-1)}) = \min_{l' \in \mathscr{L}} E(l'l^{-d/(d-1)}) \leqq E(l^{-1/(d-1)})$, on taking the particular case $l' = l$. So $E^d(l^{-1/(d-1)}) \leqq (1/m)Q(l^{-1/(d-1)}), \leqq 1/m\, E(l^{-1/(d-1)})$, so $1/m \geqq \max_{l \in \mathscr{L}} E^{d-1}(l^{-1/(d-1)})$. Hence $1/m = \max_{l \in \mathscr{L}} E^{d-1}(l^{-1/(d-1)})$, and also $1/m = \max_{l \in \mathscr{L}} Q^{d-1}(l^{-1/(d-1)})$, since $Q(l^{-1/(d-1)}) \leqq E(l^{-1/(d-1)})$.

The last equality of the theorem is obtained by noting that from Theorem 21,

$$E(l^{-1/(d-1)}) = \max_{x \in K}\left[\frac{l^{-A(x)/(d-1)}}{A(x)^{A(x)}}\right]^{1/d}.$$

Note that all the extrema above can be taken as well with l belonging to the convex set $\mathscr{L}'$.

Notice that $l^{A(e)/d} \geqq c(e)A(e)^{A(e)/d}$, so that $l^{A(x)/d} \geqq \lambda = \min_{e \in \mathscr{E}'} c(e)A(e)^{A(e)/d}$. Hence $m \geqq \lambda \min_{x \in K} A(x)^{((d-1)/d)\,A(x)}$. Since $A(x)^{A(x)}$ is no less than $(d/n)^d$ (achieved when all $A_i(x) = d/n$ if such exists), the theorem above includes incidentally Theorem 25.

If the minimum of $\mathrm{Perm}(x)$ for $x \in D_k$ is achieved at $x = f$, then Holder's inequality shows easily that $\min_{x \in D_k} \mathrm{Perm}(x^r)$, $r \geqq 1$, is also achieved at $x = f$. Our next result is a slight contribution in this direction.

THEOREM 32. *For every k there exists an r such that* $\min_{x \in D_R} \mathrm{Perm}(x^r)$ *is achieved uniquely at $x = f$.*

Let N be any neighborhood of f in D_k. If $x \in N^c$ (complement of N in D_k), then by Theorem 29 there exists an $\varepsilon > 0$, ε independent of x, and a π such that $x^\pi \geqq 1/k^k + \varepsilon$. For simplicity write $\mathrm{Perm}(x) = P(x)$. Thus, for all sufficiently large r, $P(x^r) \geqq (1/k^k + \varepsilon)^r \geqq P(f^r) = k!/k^{kr}$. So by picking r sufficiently large, we can guarantee that $\min_{x \in D_k} P(x^r)$ is achieved at a point of N.

Construct the map $x \to h(x)$, $h(x) = (1/P(x)) \sum x^\pi \pi$ as in the text following Lemma 10, with inverse map $x \to l(x)$. The critical points of $P(x^r)$ are to be found among the solutions of $h(x^r) = x$. For any $y \in R_0^n$ having F-like support for some flat F, we may write $y = \delta \gamma A(x)$, $\delta > 0$, $\gamma \in G$, $x \in \underline{F}$. Define $[y] = A(x)$. With this notation, the critical points of $P(x^r)$ are to be found among the solutions of $x = [l^{1/r}(x)]$. Now we need two lemmas.

LEMMA 33. *Let $x \in D_k$ and $\varepsilon > 0$. Suppose for every permutation matrix π, $x^{\pi/k} \geqq 1/k(1 + \varepsilon)$. Then for all i,j, $|x_{ij} - 1/k| \leqq k\varepsilon$.*

Let V be the convex set of $x \in D_k$ for which $x_{11} = a$. Let Λ be the set of permutation matrices π for which $\pi(1) = 1$. Define $F(x) = \min_{\pi \in \Lambda} x^{\pi/k}$. F is a concave function on D_k or V. If π and $\rho \in \Lambda$, note that $F(\pi \circ x \circ \rho) = F(x)$. And if $x \in V$, $\pi \circ x \circ \rho \in V$. Hence, by averaging, we see that the maximum of F on V is attained at a point x_0 for which $\pi \circ x_0 \circ \rho = x_0$, whenever $\pi, \rho \in \Lambda$. So x_0 must be

$$\begin{matrix} a & \dfrac{1-a}{k-1} & \cdots & \dfrac{1-a}{k-1} \\ \dfrac{1-a}{k-1} & \dfrac{k+a-2}{(k-1)^2} & \cdots & \dfrac{k+a-2}{(k-1)^2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \dfrac{1-a}{k-1} & \dfrac{k+a-2}{(k-1)^2} & \cdots & \dfrac{k+a-2}{(k-1)^2} \end{matrix}$$

and $F(x) \leqq a^{1/k}[(k + a - 2)/(k - 1)^2]^{(k-1)/k}$. If now x satisfies the inequalities of the lemma, then

$$\frac{1}{k(1+\varepsilon)} \leqq a^{1/k}\left[\frac{k+a-2}{(k-1)^2}\right]^{(k-1)/k} \leqq \frac{1}{k}a + \frac{k-1}{k}\,\frac{k+a-2}{(k-1)^2}$$

and thus gives immediately $a \geqq 1/k - \varepsilon$. The last inequality holds for any element x_{ij} in x. Since the row sums of x are one, we also obtain

$$1 - x_{ij} \geqq (k-1)\left(\frac{1}{k} - \varepsilon\right)$$

or

$$x_{ij} \leqq \frac{1}{k} + (k-1)\varepsilon \leqq \frac{1}{k} + k\varepsilon$$

completing proof of the lemma.

LEMMA 34. *There exists a constant* $\alpha \leqq 1/2k$ *such that if* $x \in D_k$ *and* $|x_{ij} - 1/k| \leqq \varepsilon \leqq \alpha$ *for all* i, j, *then* $|l_{ij}(x) - 1/k| \leqq \varepsilon$.

Let u be a matrix with row and column sums zero, t a real variable. The Taylor series for $h(f + tu)$ starts as

$$h(f + t_u) = f + t\frac{k}{k-1}u + \cdots .$$

So the Taylor series for $l(f + tu)$ has the form

$$l(f + t_u) = f + t\frac{k-1}{k}u + \cdots .$$

For $t = 1$, the series converges for all sufficiently small u, and the lemma is clear.

Now we complete the proof of Theorem 32. Pick r so large that $\min_{x \in D_k} P(x)$ is achieved at a point x belonging to $N = \{x \mid |x_{ij} - 1/k| \leqq \alpha\}$, with α of Lemma 34, and also demand $r \geqq 8k^2$.

Let $x \in N$, and suppose $|x_{ij} - 1/k| \leqq \varepsilon \leqq \alpha$, for all i, j. Then $|l_{ij}(x) - 1/k| \leqq \varepsilon$, so for any permutations π and ρ, $l^{\pi/k}/l^{\rho/k} \leqq (1 + k\varepsilon)/(1 - k\varepsilon) \leqq 1 + 4k\varepsilon$. Put $y = [l^{1/r}(x)]$. Then $y^{\pi/k}/y^{\rho/k} = l^{\pi/kr}/l^{\rho/kr} \leqq (1 + 4k\varepsilon)^{1/r} = (1 + 4k\varepsilon)^{1/r} \cdot 1^{r-1/r}$ $\leqq (1/r)(1 + 4k\varepsilon) + (r-1)/r = (1 + (4k/r)\varepsilon)$. Since there exists a permutation ρ such that $y^{\rho} \geqq 1/k^k$, we obtain for all π, $y^{\pi/k} \geqq 1/k(1 + (4k/r)\varepsilon)$. Then by Lemma 33, $|y_{ij} - 1/k| \leqq (4k^2/r)\varepsilon$, and since we have selected $r \geqq 8k^2$, we have

$$|y_{ij} - 1/k| \leqq \frac{1}{2}\varepsilon.$$

If now x is a fixed point of map $x \to [l^{1/r}(x)]$, then $x_{ij} - 1/k$ is arbitrarily small; that is, $x = f$, and we are done.

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